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Elongated U(1) instantons on noncommutative $\mathbb{R}^4$

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Abstract: We found an exact solution of elongated U(1) instanton on noncommutative $\mathbb{R}^4$ for general instanton number k. The deformed ADHM equation was solved with general k and the gauge connection and the curvature were given explicitly. We also checked our solutions and evaluated the instanton charge by a numerical calculation.

Keywords: Solitons Monopoles and Instantons, Non-Commutative Geometry.

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Contents

1. Introduction

Recently, noncommutative geometry and noncommutative field theory have been revived in the string theory and are studied actively [1, 2, 3]. In this concern, noncommutative instantons have been discussed for recent three years. Especially U(1) gauge group case is analyzed in detail and there are some concrete form of gauge field whose curvature is (anti-)self-dual. However, these explicit form is given only at the case of instanton number $k = 1$ and 2 [4, 5, 6]. The gauge field of a general k instanton had never been constructed for the reason of the difficulty of the calculation without some special cases. For example, when the noncommutative parameter is given as anti-self-dual, then the solutions of the anti-self-dual equation of the curvature are obtained by simple solution generating technique [7, 8, 9]. These solutions (called localized instanton) correspond to the case that the stability condition of moduli space is removed [10]. Therefore we have to construct an explicit expression of the instanton satisfying stability condition.[1] If an exact expression of a such gauge field

[1]In the Braden and Nekrasov'spaper [14], they give the elongated instanton solution for general k as the solution of deformed ADHM equations. The solution contains the Charlier polynomials, and corresponds to commutative space instanton with nontrivial metric. On the other hand, our elongated instantons are given in noncommutative $\mathbb{R}^4$ (section 3). Therefore our solutions are different from their one.

and its curvature are given, they will be powerful tool for many kind of calculations. For example, in cohomological Yang-Mills theory we had to evaluate sum of Euler number of instanton moduli space for each instanton number. In the similar case of noncommutative Yang-Mills, the explicit expression of gauge field is expected to be a more powerful tool since we have similar example in GMS solitons [11, 12, 13].

In this paper, noncommutative U(1) instantons with arbitary instanton number k are constructed with deformed ADHM procedure. The gauge connections and curvature 2-form are given as a concrete form with number operator (Fock space) representation at the same time.

Another motivation of this paper is to resolve the issue that instanton number in noncommutative ADHM construction do not correspond to the integral of the first Pontrjagin class (instanton charge) with clear way, where we call the size of matrices in ADHM construction instanton number. Recently, some new results about the instanton number of the noncommutative $k = 2$ U(1) instanton is given [5, 6]. We also analyze higher instanton number case in section 4 by a numerical way.

2. Noncommutative U(1) instantons

2.1 Noncommutative $\mathbb{R}^4$ and the Fock space representation

We consider euclidean noncommutative $\mathbb{R}^4$, whose coordinates x^μ ($\mu = 1, 2, 3, 4$) satisfy the commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}\,, \tag{2.1}$$

where $\theta^{\mu\nu}$ is an antisymmetric real constant matrix, and is called noncommutative parameter. We can always bring $\theta^{\mu\nu}$ to the skew-diagonal form

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \theta^{12} & 0 & 0 \\ -\theta^{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta^{34} \\ 0 & 0 & -\theta^{34} & 0 \end{pmatrix} \tag{2.2}$$

by space rotation. For simplicity, we restrict the noncommutativity of space to the case of $\theta^{12} = \theta^{34} = -\zeta$ ($\zeta > 0$). Here we introduce complex coordinates

$$z_1 = \frac{1}{\sqrt{2}}(x^1 + ix^2)\,, \qquad z_2 = \frac{1}{\sqrt{2}}(x^3 + ix^4)\,, \tag{2.3}$$

then the commutation relations (2.1) become

$$[z_1, \bar{z}_1] = [z_2, \bar{z}_2] = -\zeta\,, \qquad \text{others are zero}\,. \tag{2.4}$$

For using the usual operator representation, we define creation and annihilation operators by

$$c_\alpha^\dagger = \frac{z_\alpha}{\sqrt{\zeta}}\,, \quad c_\alpha = \frac{\bar{z}_\alpha}{\sqrt{\zeta}}\,, \quad [c_\alpha, c_\alpha^\dagger] = 1 \qquad (\alpha = 1, 2)\,. \tag{2.5}$$

The Fock space $\mathcal{H}$ on which the creation and annihilation operators (2.5) act is spanned by the Fock states

$$|n_1, n_2\rangle = \frac{(c_1^\dagger)^{n_1}(c_2^\dagger)^{n_2}}{\sqrt{n_1!n_2!}} |0,0\rangle\,, \tag{2.6}$$

with

$$\begin{aligned} c_1\,|n_1,n_2\rangle &= \sqrt{n_1}\,|n_1-1,n_2\rangle\,, \qquad & c_1^\dagger\,|n_1,n_2\rangle &= \sqrt{n_1+1}\,|n_1+1,n_2\rangle\ , \\ c_2\,|n_1,n_2\rangle &= \sqrt{n_2}\,|n_1,n_2-1\rangle\,, \qquad & c_2^\dagger\,|n_1,n_2\rangle &= \sqrt{n_2+1}\,|n_1,n_2+1\rangle\ , \end{aligned} \tag{2.7}$$

where n_1 and n_2 are the occupation number. The number operators are also defined by

$$\hat{n}_\alpha = c_\alpha^\dagger c_\alpha\,, \qquad \hat{N} = \hat{n}_1 + \hat{n}_2\,, \tag{2.8}$$

which act on the Fock states as

$$\hat{n}_\alpha\,|n_1,n_2\rangle = n_\alpha\,|n_1,n_2\rangle\ , \qquad \hat{N}\,|n_1,n_2\rangle = (n_1+n_2)\,|n_1,n_2\rangle\ . \tag{2.9}$$

In the operator representation, derivatives of a function $f(z_1,\bar{z}_1,z_2,\bar{z}_2)$ are defined by

$$\partial_\alpha f(z) = \frac{1}{\zeta}[\bar{z}_\alpha, f(z)]\,, \qquad \bar{\partial}_\alpha f(z) = -\frac{1}{\zeta}[z_\alpha, f(z)]\,. \tag{2.10}$$

The integral on noncommutative $\mathbb{R}^4$ is defined by the standard trace in the operator representation,

$$\int d^4x = \int d^4z = (2\pi\zeta)^2\,\mathrm{Tr}\ . \tag{2.11}$$

Note that Tr represents the trace over the Fock space whereas the trace over the gauge group is represented by tr.

2.2 Noncommutative gauge theory and instantons

Here we consider a $\mathrm{U}(N)$ Yang-Mills theory on noncommutative $\mathbb{R}^4$.

In the noncommutative space, the Yang-Mills connection is defined as [4, 15]

$$\nabla_\mu \Psi = i\Psi\theta_{\mu\nu}x^\nu + D_\mu\Psi\,, \tag{2.12}$$

where Ψ is matter fields and D_μ is the gauge field which is defined as anti-hermitian. The Yang-Mills curvature of the connection ∇_μ is

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = -i\theta_{\mu\nu} + [D_\mu, D_\nu]\,. \tag{2.13}$$

In our notation of the complex coordinates (2.3) and (2.4), the curvatures (2.13) are

$$F_{\alpha\bar{\alpha}} = \frac{1}{\zeta} + [D_\alpha, \bar{D}_{\bar{\alpha}}], \quad F_{\alpha\bar{\beta}} = [D_\alpha, \bar{D}_{\bar{\beta}}] \qquad (\alpha \neq \beta)\,. \tag{2.14}$$

JHEP11(2001)068

The Yang-Mills action is given by

$$S = -\frac{1}{g^2}\int \mathrm{tr}_N F \wedge *F\,, \tag{2.15}$$

where tr_N represents a trace for the gauge group U(N), g is the Yang-Mills coupling and $*$ is the Hodge-star operation. The equation of motion is

$$[\nabla_\mu, F_{\mu\nu}] = 0\,. \tag{2.16}$$

(Anti-)instanton solutions are special solutions of (2.16) which satisfy the (anti)-self-duality condition

$$F = \pm * F\,. \tag{2.17}$$

(Anti-)self-duality conditions in the complex coordinates are

$$F_{1\bar{1}} = +\,F_{2\bar{2}}\,,\quad F_{1\bar{2}} = F_{\bar{1}2} = 0 \qquad (\text{self-dual})\,, \tag{2.18}$$

$$F_{1\bar{1}} = -\,F_{2\bar{2}}\,,\quad F_{12} = F_{\bar{1}\bar{2}} = 0 \qquad (\text{anti-self-dual})\,. \tag{2.19}$$

In the commutative space, these solutions are classified by the topological charge

$$Q = -\frac{1}{8\pi^2}\int \mathrm{tr}_N F \wedge F\,, \tag{2.20}$$

which is always integer and is called instanton number k. However, in the noncommutative space this statement is unclear. We discuss this issue in section 4 by using the operator representation of (2.20):

$$Q = \begin{cases} -\zeta^2\,\mathrm{Tr}\,\mathrm{tr}_N(F_{1\bar{1}}F_{\bar{1}1} + F_{12}F_{\bar{1}\bar{2}}) & (\text{self-dual}) \\ \zeta^2\,\mathrm{Tr}\,\mathrm{tr}_N(F_{1\bar{1}}F_{\bar{1}1} + F_{1\bar{2}}F_{\bar{1}2}) & (\text{anti-self-dual})\,. \end{cases} \tag{2.21}$$

2.3 Nekrasov-Schwarz noncommutative U(1) instantons

In the ordinary commutative space, there is a well-known way to find (anti)-self-dual configurations of the gauge fields. It is the ADHM construction which is proposed by Atiyah, Drinfeld, Hitchin and Manin [16]. Nekrasov and Schwarz first extended this method to noncommutative space [17]. Especially U(1) case is discussed in [14, 4, 18, 19] in detail.

In commutative space case, the U(1) instanton is impossible to exist. However, in noncommutative space case, nontrivial U(1) instantons exist. Here we show a brief review on the ADHM construction of U(1) instanton [4].

The first step of the ADHM construction is looking for matrices B_1, B_2, I and J which satisfy the deformed ADHM equations

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 2\zeta, \tag{2.22}$$

$$[B_1, B_2] + IJ = 0\,, \tag{2.23}$$

where B_1 and B_2 are $k \times k$ complex matrices, I and $J^\dagger$ are $k \times 1$ complex matrices. In U(1) case, if $\zeta > 0$, eq. (2.23) and the stability condition allows us to take $J = 0$ [20, 19]. In [4], a projector which projects the Hilbert space $\mathcal{H}$ to the subspace of $\mathcal{H}$ is given as

$$P = I^\dagger e^{\sum_\alpha \beta_\alpha^\dagger c_\alpha^\dagger} |0,0\rangle\, G^{-1} \langle 0,0|\, e^{\sum_\alpha \beta_\alpha c_\alpha} I, \tag{2.24}$$

where $B_\alpha = \sqrt{\zeta}\beta_\alpha$ and G is a normalization factor (hermitian matrix)

$$G = \langle 0,0|\, e^{\sum_\alpha \beta_\alpha c_\alpha} I I^\dagger e^{\sum_\alpha \beta_\alpha^\dagger c_\alpha^\dagger} |0,0\rangle\,. \tag{2.25}$$

We introduce a shift operator S which satisfies

$$SS^\dagger = 1, \qquad S^\dagger S = 1 - P\,. \tag{2.26}$$

Using the shift operator S, the U(1) anti-self-dual gauge fields are given as

$$D_\alpha = \sqrt{\frac{1}{\zeta}} S \Lambda^{-\frac{1}{2}} c_\alpha \Lambda^{\frac{1}{2}} S^\dagger\,, \qquad \bar{D}_{\bar{\alpha}} = -\sqrt{\frac{1}{\zeta}} S \Lambda^{\frac{1}{2}} c_\alpha^\dagger \Lambda^{-\frac{1}{2}} S^\dagger\,, \tag{2.27}$$

where

$$\Lambda = 1 + I^\dagger \hat{\Delta}^{-1} I\,, \qquad \hat{\Delta} = \zeta \sum_\alpha (\beta_\alpha - c_\alpha^\dagger)(\beta_\alpha^\dagger - c_\alpha)\,. \tag{2.28}$$

In the following section, we construct an explicit form of the gauge fields (2.27).

3. Construction of elongated U(1) instantons

We know the ADHM construction, but in usual, it is hardly difficult to write down explicitly the solutions of instantons for arbitrary instanton number k even in the U(1) case. However, in the case of the elongated U(1) instantons, we can get the explicit expression for general k instanton.

Let matrices B_1, B_2, I and J be

$$B_1 = \sum_{i=1}^{k-1} \sqrt{2i\zeta}\, e_i e_{i+1}^\dagger\,, \qquad B_2 = 0\,, \qquad I = \sqrt{2k\zeta}\, e_k\,, \qquad J = 0\,, \tag{3.1}$$

where e_i is defined as

$$e_i^\dagger = (\overset{1}{0}, \ldots, \overset{i-1}{0}, \overset{i}{1}, \overset{i+1}{0}, \ldots, \overset{k}{0})\,. \tag{3.2}$$

These matrices (3.1) satisfy the deformed ADHM equations (2.22) and (2.23). These ADHM data corresponds to the configuration that k instantons are elongated into z_1-$\bar{z}_1$ direction. Next step to get the explicit expression of the gauge field (2.27) is to find the explicit form of Λ in (2.28). In our case, $\hat{\Delta}$ in (2.28) is

$$\hat{\Delta} = \zeta \left[\hat{N} + \sum_{i=1}^{k-1} \left\{ 2i e_i e_i^\dagger - \sqrt{2i}(c_1 e_i e_{i+1}^\dagger + c_1^\dagger e_{i+1} e_i^\dagger) \right\} \right], \tag{3.3}$$

and Λ is

$$\Lambda = 1 + 2k\zeta\hat{\Delta}^{-1}_{kk}\,, \tag{3.4}$$

where $\hat{\Delta}^{-1}_{kk}$ is the (k,k) component of the matrix $\hat{\Delta}^{-1}$. It is sufficient for getting $\hat{\Delta}^{-1}_{kk}$ that we know the k-th line of $\hat{\Delta}^{-1}$ which is defined by $\hat{\Delta}^{-1}\hat{\Delta} = 1$. For this purpose, we define a vector $\hat{\mathbf{u}}$ as

$$\hat{\mathbf{u}}^t = \frac{1}{\zeta}\sum_{i=1}^{k}\hat{u}_i e_i^\dagger\,, \tag{3.5}$$

and following calculation is performed:

$$\begin{aligned}\hat{\mathbf{u}}^t\hat{\Delta} = &\left\{\hat{u}_1(\hat{N}+2) - \hat{u}_2 c_1^\dagger\right\} e_1^\dagger + \\ &+ \sum_{i=2}^{k-1}\left\{-\sqrt{2(i-1)}\hat{u}_{i-1}c_1 + \hat{u}_i(\hat{N}+2i) - \sqrt{2i}\hat{u}_{i+1}c_1^\dagger\right\} e_i^\dagger + \\ &+ \left\{-\sqrt{2(k-1)}\hat{u}_{k-1}c_1 + \hat{u}_k\hat{N}\right\} e_k^\dagger\,.\end{aligned} \tag{3.6}$$

To satisfy the equation of k-th line of $\hat{\Delta}^{-1}\hat{\Delta} = 1$, the recurrence relation is imposed:

$$\begin{aligned}\hat{u}_2 c_1^\dagger - \hat{u}_1(\hat{N}+2) &= 0\,, \\ \sqrt{2i}\hat{u}_{i+1}c_1^\dagger - \hat{u}_i(\hat{N}+2i) + \sqrt{2(i-1)}\hat{u}_{i-1}c_1 &= 0 \qquad (2 \leq i \leq k-1)\,.\end{aligned} \tag{3.7}$$

Then we obtain $\hat{\Delta}^{-1}_{kk}$ as

$$\hat{\Delta}^{-1}_{kk} = \frac{1}{\zeta}\left\{-\sqrt{2(k-1)}\hat{u}_{k-1}c_1 + \hat{u}_k\hat{N}\right\}^{-1}\hat{u}_k\,. \tag{3.8}$$

Now we substitute $\hat{w}_i(\hat{n}_1,\hat{n}_2)$ for $\hat{u}_i$:

$$\hat{u}_i = \hat{w}_{i-1}(\hat{n}_1,\hat{n}_2)\frac{(c_1^\dagger)^{k-i}}{\sqrt{2^{i-1}(i-1)!}}\,, \tag{3.9}$$

and rewrite the recurrence relation (3.7) as

$$\begin{aligned}w_1 - (N-k+3)w_0 &= 0\,, \\ w_{i+1} - (N+3i-k+3)w_i + 2i(n_1+i-k+1)w_{i-1} &= 0 \\ &(1 \leq i \leq k-2)\,.\end{aligned} \tag{3.10}$$

Note that $w_i(n_1,n_2)$ depends on only number operators, then its inverse is formally given by $w_i^{-1}(n_1,n_2)$, where n_1 and n_2 are replaced by their eigenvalues. We can fortunately solve the recurrence relation (3.10). The generating function of w_i is

$$F(t) = (1-t)^{-n_1+n_2+k-1}(1-2t)^{-n_2-1} = \sum_{i=0}^{\infty}\frac{w_i}{i!}t^i\,. \tag{3.11}$$

Then $w_i(n_1,n_2)$ is given as

$$w_i(n_1,n_2) = \left(\frac{d}{dt}\right)^i F(t)\bigg|_{t=0} . \tag{3.12}$$

Using this $w_i(n_1,n_2)$, we can write $\Lambda(n_1,n_2)$ as

$$\Lambda(n_1,n_2) = \frac{w_k(n_1,n_2)}{w_k(n_1,n_2) - 2k w_{k-1}(n_1,n_2)} . \tag{3.13}$$

Next step, we should determine a shift operator. In our case the projector (2.24) is

$$P = \sum_{n_1=0}^{k-1} |n_1,0\rangle \langle n_1,0| \ , \tag{3.14}$$

then we can define the shift operator as

$$S^\dagger = \sum_{n_1=0}^{\infty} |n_1+k,0\rangle \langle n_1,0| + \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty} |n_1,n_2\rangle \langle n_1,n_2| \ . \tag{3.15}$$

Using the above results, we can write down the elongated U(1) instanton gauge fields on noncommutative $\mathbb{R}^4$ explicitly:

$$D_1 = \sqrt{\frac{1}{\zeta}} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} |n_1,n_2\rangle \langle n_1+1,n_2| \, d_1(n_1,n_2;k) \, , \tag{3.16}$$

$$\bar{D}_{\bar{1}} = -\sqrt{\frac{1}{\zeta}} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} |n_1+1,n_2\rangle \langle n_1,n_2| \, d_1(n_1,n_2;k) \, , \tag{3.17}$$

$$\begin{aligned} D_2 = \sqrt{\frac{1}{\zeta}} \Bigg\{ &\sum_{n_1=0}^{\infty} |n_1,0\rangle \langle n_1+k,1| \, d_2(n_1,0;k) + \\ &+ \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty} |n_1,n_2\rangle \langle n_1,n_2+1| \, d_2(n_1,n_2;k) \Bigg\} , \end{aligned} \tag{3.18}$$

$$\begin{aligned} \bar{D}_{\bar{2}} = -\sqrt{\frac{1}{\zeta}} \Bigg\{ &\sum_{n_1=0}^{\infty} |n_1+k,1\rangle \langle n_1,0| \, d_2(n_1,0;k) + \\ &+ \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty} |n_1,n_2+1\rangle \langle n_1,n_2| \, d_2(n_1,n_2;k) \Bigg\} , \end{aligned} \tag{3.19}$$

where

$$d_1(n_1,n_2;k) = \begin{cases} \sqrt{n_1+k+1} \left[\frac{\Lambda(n_1+k+1,0)}{\Lambda(n_1+k,0)}\right]^{\frac{1}{2}} & (n_2 = 0), \\ \sqrt{n_1+1} \left[\frac{\Lambda(n_1+1,n_2)}{\Lambda(n_1,n_2)}\right]^{\frac{1}{2}} & (n_2 \neq 0), \end{cases} \tag{3.20}$$

$$d_2(n_1,n_2;k) = \begin{cases} \left[\frac{\Lambda(n_1+k,1)}{\Lambda(n_1+k,0)}\right]^{\frac{1}{2}} & (n_2 = 0), \\ \sqrt{n_2+1} \left[\frac{\Lambda(n_1,n_2+1)}{\Lambda(n_1,n_2)}\right]^{\frac{1}{2}} & (n_2 \neq 0). \end{cases} \tag{3.21}$$

Its curvature is presented in appendix A.

4. Numerical check and analysis

In section 3, we have constructed the solution of the elongated U(1) instantons with general instanton number k. In this section we check and analyze our solution numerically.

4.1 Anti-self-duality

Since our solution would be an anti-self-dual configuration, it should satisfy the condition (2.19). We can show that the condition $F_{12} = F_{\bar{1}\bar{2}} = 0$ for every component $|n_1, n_2\rangle \langle n_1, n_2|$, and $F_{1\bar{1}} = -F_{2\bar{2}}$ on $|0,0\rangle \langle 0,0|$ component are satisfied with analytic calculation easily. However, it is difficult to check the condition $F_{1\bar{1}} = -F_{2\bar{2}}$ for any component analytically. Therefore we check the condition numerically and the results satisfy (2.19) completely.

4.2 Instanton number and density

In this subsection, we give the numerical study on our instanton solution. The instanton charge (2.21) is given by the infinite series; the trace operation is performed by the sum over the Fock states $|n_1, n_2\rangle$, or the space-time points (n_1, n_2). Since the sum contains the infinite points, in the numerical analysis we restrict the number of space-time points, that is, we put the cut-off number for the Infrared scale. Then we define the instanton charge with the cut-off number n;

$$Q_n = -k \sum_{n_1=0}^{n} \sum_{n_2=0}^{n} \rho(n_1, n_2)\,, \tag{4.1}$$

$$Q_\infty = \lim_{n\to 0} Q_n = Q\,, \tag{4.2}$$

and we define the instanton density as

k	Q_n $n=10$	$n=20$	$n=50$
1	-0.991033	-0.997442	-0.999557
2	-1.96443	-1.98977	-1.99823
3	-2.91641	-2.97653	-2.99599
4	-3.83934	-3.95662	-3.99282
5	-4.72416	-4.92835	-4.98864
6	-5.56423	-5.88927	-5.98336
7	-6.35653	-6.83624	-6.97688
8	-7.09772	-7.76553	-7.96905
9	-7.79167	-8.67337	-8.95968
10	-8.45736	-9.55689	-9.94857
15	-11.1219	-13.6201	-14.855

Table 1: Numerical check of Q (i): k is the instanton number. We summed over the Fock space up to $(n_1, n_2) = (10,10), (20,20), (50,50)$.

$$\rho(n_1, n_2) := -\frac{\zeta^2}{k} \langle n_1, n_2| \left(F_{1\bar{1}}F_{\bar{1}1} + F_{1\bar{2}}F_{\bar{1}2}\right) |n_1, n_2\rangle\ . \tag{4.3}$$

The numerical results of the instanton charge is listed in the table 1. At each instanton solution we give the three results; the sum up to the cut-off number $n = 10, 20$ and 50. These results show that if we summed over the Fock space up to a higher number n, the numerical calculation of the instanton charge approaches the expected number.

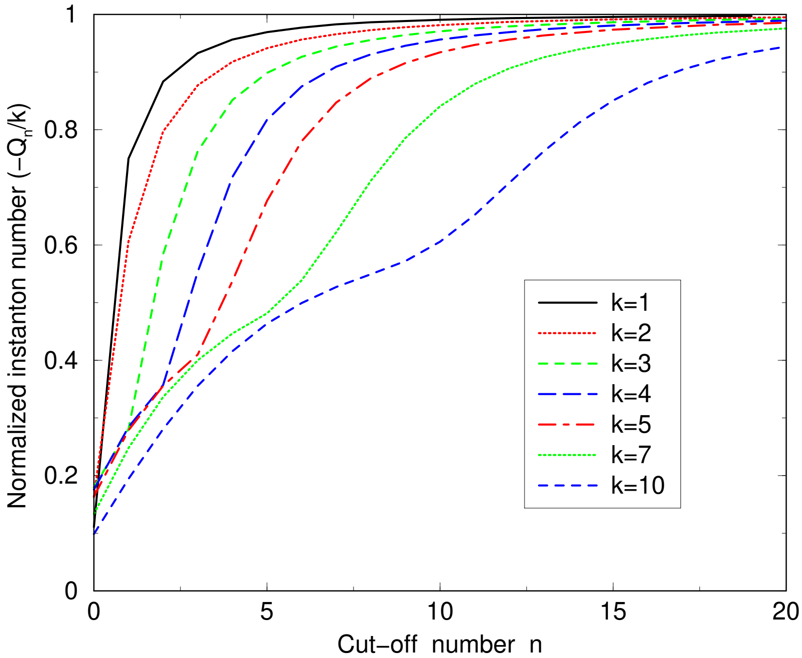

Figure 1: Convergency of the instanton charge; As the cut-off number n increases, all the normalized instanton number $-Q_n/k$ approaches one. These results show that our solution is suitable and gives the instanton number precisely.

Figure 1 shows the convergence of the numerical calculation of the instanton charge. In this figure we find that each normalized instanton number $(-Q_n/k)$ approaches to one as we take a higher cut-off number n. Then we conclude the convergency of the instanton charge of our solution is suitable. The table 2 shows the cut-off number that gives 99% of the expected value. This 99% cut-off number increases as the instanton charge increases. This results mean that a high number instanton solution is expanded into the space direction (n_1, n_2). The distributions of the instanton density are given in figure 2–8. The figure 2 show the instanton density of the one instanton solution. In this figure we find that the configuration near the origin dominates (the maximum point is $(n_1, n_2) = (0, 1)$). As the instanton number increase the configuration near the origin spreads out into the n_1 direction; the point giving the extreme value moves along the $n_2 = 1$ line, and the new maximum point appears from the origin (figure 6–8). Our instanton solutions do not expand into the n_2 direction and are localized around the n_2=0 plane ($|z_2| = 0$ plane).

k	n of 99% line
1	10
2	14
3	18
4	21
5	24
6	27
7	30
8	33
9	35
10	38
15	50

Table 2: Numerical check of Q (ii): n of 99% line denotes the cut-off number n at which $-Q_n/k \simeq 0.99$.

JHEP11(2001)068

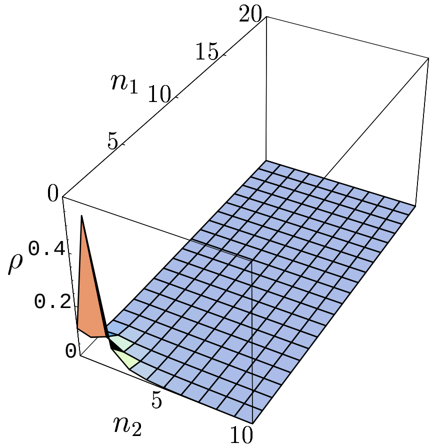

Figure 2: $k = 1$ instanton-density; The height of the graph is the instanton density. The instanton density is localized around the maximum point $(n_1, n_2) = (0, 1)$. At this maximum point the instanton density is $\rho = 0.55556$.

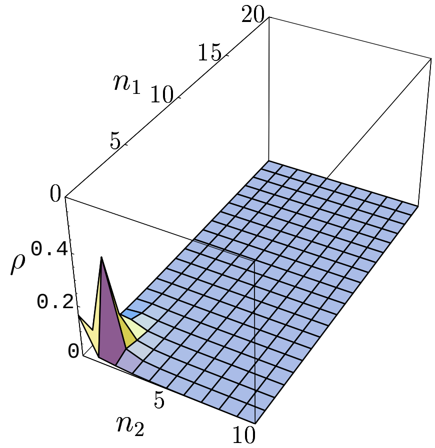

Figure 3: $k = 2$ instanton density; At the point $(n_1, n_2) = (1, 1)$ the instanton density takes the maximum value $\rho = 0.366429$.

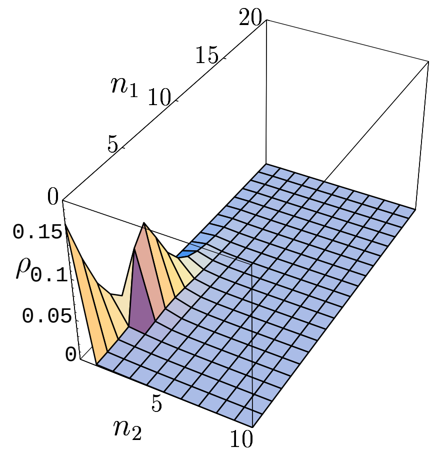

Figure 4: $k = 5$ instanton density; At the origin the instanton density takes the maximum value $\rho = 0.163719$. Another extreme value exists at the point $(n_1, n_2) = (5, 1)$, and is $\rho = 0.109048$.

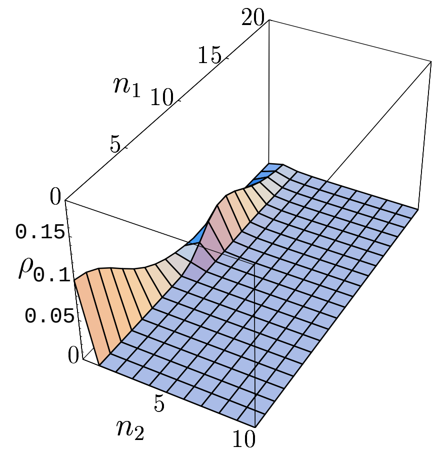

Figure 5: $k = 10$ instanton density; At the origin the instanton density takes the maximum value $\rho = 0.0989281$. Another extreme value is $\rho = 0.0449079$ at the point $(n_1, n_2) = (13, 1)$.

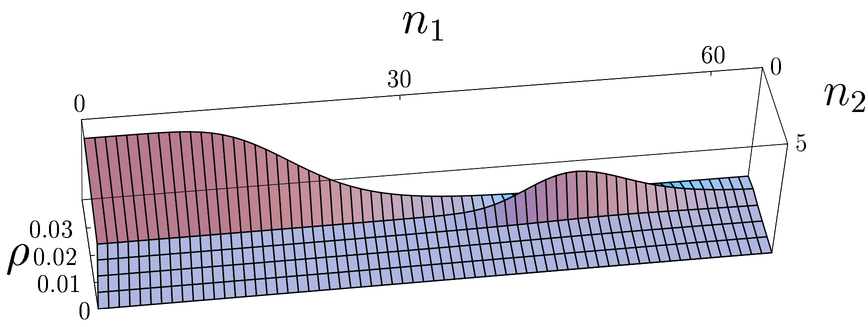

Figure 6: $k = 30$ instanton density; At the origin the instanton density takes the maximum value $\rho = 0.0333333$. And the another extreme value is $\rho = 0.0127042$ at the point $(n_1, n_2) = (46, 1)$.

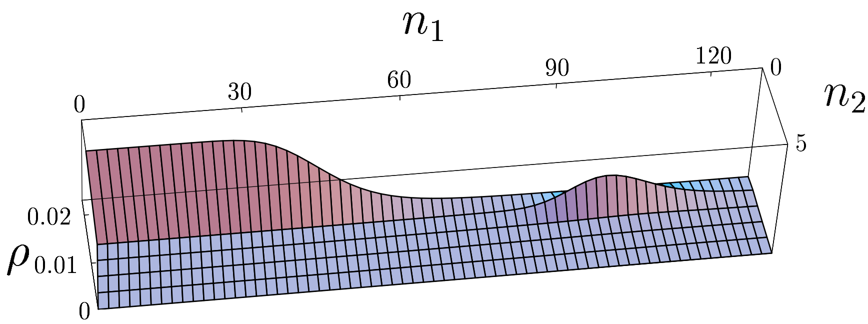

Figure 7: $k = 60$ instanton density: On the $n_2 = 0$ line, the instanton density near the origin takes the maximum value $\rho = 0.0166667$. And the another extreme value is $\rho = 0.00610433$ at the point $(n_1, n_2) = (101, 1)$.

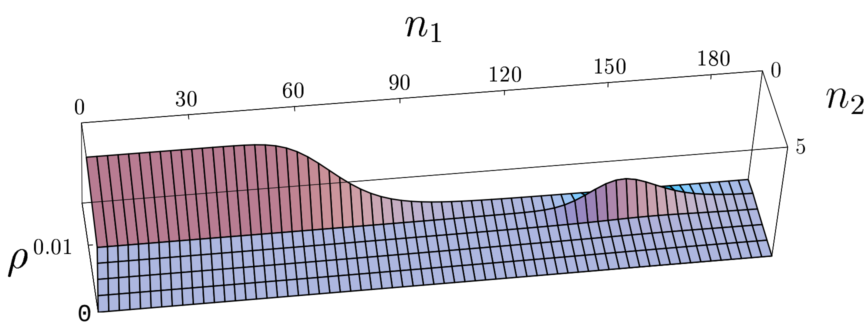

Figure 8: $k = 90$ instanton density: On the $n_2 = 0$ line the instanton density near the origin takes the maximum value $\rho = 0.0111111$. And the another extreme value is $\rho = 0.00397892$ at the point $(n_1, n_2) = (157, 1)$.

5. Discussion and conclusion

Explicit expression of the elongated U(1) instanton on noncommutative $\mathbb{R}^4$ was obtained for general instanton number. Deformed ADHM construction was used there, and an important progress is to construct exact form of the gauge field and the curvature. As a result of this, we could perform two analysis of geometrical and topological nature with numerical way. First, the anti-self-dual condition is confirmed. Second, numerical calculation of the instanton charge that defined by eq. (2.20) was done, and it implies that both instanton charge Q defined by eq. (2.20) and instanton number defined by the size k of the matrices of ADHM construction is equivalent. Additionally, we saw the distribution of instanton density of the elongated instanton and it is indeed elongated to the z_1-$\bar{z}_1$ direction. From the point of view of Hilbert scheme, as Nakajima and Furuuchi said in [20, 19], B_i determine the ideal. In our case we take $B_2 = 0$ and $B_1 \neq 0$, and off-diagonal part of B_1 is regarded as the trace of k-points elongated to z_1-$\bar{z}_1$ direction before shrinking into the origin. This effect is still alive in density of the topological charge.

In these investigation, some geometrical natures of noncommutative space appear. For example, some kind of recurrence relations plays a role of a differential equation on usual commutative spaces, i.e., local geometry is written by some series. Therefore the integration of some kind of topological charge is replaced by the sum of series. Unfortunately, many of topological charge are difficult to calculate by analytic way. This fact will demand to review the theory of sequence from a quantum geometrical point of view.

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A. Curvature of our solution

We present the curvature of the elongated U(1) instanton with general instanton number k by using the connection (3.19). Our solution is an anti-self-dual configuration, so $F_{12} = F_{\bar{1}\bar{2}} = 0$. The other components are

$$F_{1\bar{1}} = \frac{1}{\zeta} - \frac{1}{\zeta}\left[\sum_{n_2=0}^{\infty} |0,n_2\rangle\langle 0,n_2| \big(d_1(0,n_2;k)\big)^2 + \right. \tag{A.1}$$
$$\left. + \sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty} |n_1,n_2\rangle\langle n_1,n_2| \left\{\big(d_1(n_1,n_2;k)\big)^2 - \big(d_1(n_1-1,n_2;k)\big)^2\right\}\right],$$

$$F_{2\bar{2}} = \frac{1}{\zeta} - \frac{1}{\zeta}\Bigg[\sum_{n_1=0}^{\infty} |n_1,0\rangle\langle n_1,0| \big(d_2(n_1,0;k)\big)^2 + \sum_{n_1=0}^{k-1} |n_1,1\rangle\langle n_1,1| \big(d_2(n_1,1;k)\big)^2 +$$
$$+ \sum_{n_1=k}^{\infty} |n_1,1\rangle\langle n_1,1| \Big\{\big(d_2(n_1,1;k)\big)^2 - \big(d_2(n_1-k,0;k)\big)^2\Big\} +$$
$$+ \sum_{n_1=0}^{\infty}\sum_{n_2=2}^{\infty} |n_1,n_2\rangle\langle n_1,n_2| \Big\{\big(d_2(n_1,n_2;k)\big)^2 - \big(d_2(n_1,n_2-1;k)\big)^2\Big\}\Bigg], \tag{A.2}$$

$$F_{1\bar{2}} = -\frac{1}{\zeta}\Bigg[|k-1,1\rangle\langle 0,0|\, d_1(k-1,1;k)d_2(0,0;k) +$$
$$+ \sum_{n_1=1}^{\infty} |n_1+k-1,1\rangle\langle n_1,0| \Big\{d_1(n_1+k-1,1;k)d_2(n_1,0;k) -$$
$$- d_1(n_1-1,0;k)d_2(n_1-1,0;k)\Big\}$$
$$+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} |n_1-1,n_2+1\rangle\langle n_1,n_2| \Big\{d_1(n_1-1,n_2+1;k)d_2(n_1,n_2;k) -$$
$$- d_1(n_1-1,n_2;k)d_2(n_1-1,n_2;k)\Big\}\Bigg], \tag{A.3}$$

$$F_{\bar{1}2} = -F_{1\bar{2}}^{\dagger}$$
$$= \frac{1}{\zeta}\Bigg[|0,0\rangle\langle k-1,1|\, d_1(k-1,1;k)d_2(0,0;k) +$$
$$+ \sum_{n_1=1}^{\infty} |n_1,0\rangle\langle n_1+k-1,1| \Big\{d_1(n_1+k-1,1;k)d_2(n_1,0;k) -$$
$$- d_1(n_1-1,0;k)d_2(n_1-1,0;k)\Big\}$$
$$+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} |n_1,n_2\rangle\langle n_1-1,n_2+1| \Big\{d_1(n_1-1,n_2+1;k)d_2(n_1,n_2;k) -$$
$$- d_1(n_1-1,n_2;k)d_2(n_1-1,n_2;k)\Big\}\Bigg], \tag{A.4}$$

where $d_1(n_1,n_2;k)$ and $d_2(n_1,n_2;k)$ are defined by (3.20) and (3.21). At a first sight, it is difficult to check whether the condition $F_{1\bar{1}} = -F_{2\bar{2}}$ is satisfied or not. However, we confirmed that the condition is satisfied really by using a numerical evaluation.

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